Sec. 14.6:

Directional Derivatives and the Gradient Vector

What We Will Go Over In Section 14.6

1. The Gradient Vector
2. Directional Derivatives
3. Maximizing the Directional Derivative
4. Tangent Planes to Level Surfaces

1. The Gradient Vector

Def: Let $f : D \subseteq \mathbb{R}^2 \to \mathbb{R}$ be a function of 2 variables.
The gradient of f (denoted ∇f) is the vector…

$$\nabla f \equiv < \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} >$$

Note: In this case, ∇f is a function from $\mathbb{R}^2$ to V_2

Def: Let $f : D \subseteq \mathbb{R}^3 \to \mathbb{R}$ be a function of 3 variables.
The gradient of f (denoted ∇f) is the vector…

$$\nabla f \equiv < \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} >$$

Note: In this case, ∇f is a function from $\mathbb{R}^3$ to V_3

1. The Gradient Vector

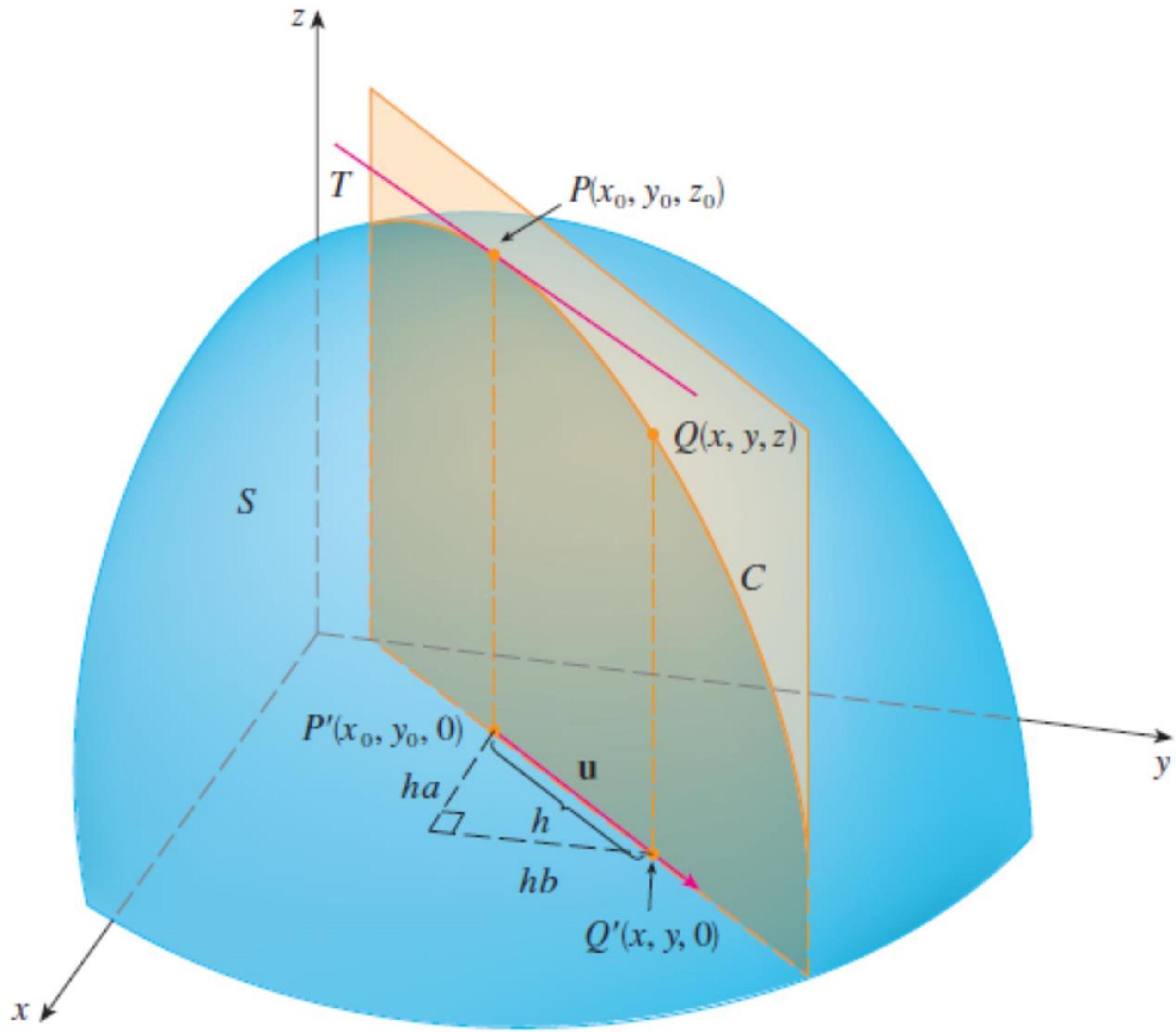
Ex 1: If $f(x,y) = x^2y^5 + \ln(xy)$ and $g(x,y,z) = \frac{z^2}{x} + xe^y$. Find…

a) ∇f

b) ∇g

2. Directional Derivatives

<u>Idea</u>: Given the graph of a 2-variable function and a point …



2. Directional Derivatives

Def: The directional derivative of f at (x_0, y_0) in the direction of a unit vector $\vec{u} =< a, b >$ is…

$$D_{\vec{u}} f(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h}$$

if this limit exists.

2. Directional Derivatives

Thm: If f is a differentiable function of x and y, then f has a directional derivative in the direction of any unit vector $\vec{u} = < a, b >$ and …

$$D_{\vec{u}} f(x, y) = \nabla f(x, y) \cdot \vec{u}$$

$$D_{\vec{u}} f(x_0, y_0) = \nabla f(x_0, y_0) \cdot \vec{u}$$

Note: $\vec{u}$ must be a unit vector. If it isn't, divide by its magnitude to get a vector in the same direction as $\vec{u}$ that IS a unit vector.

2. Directional Derivatives

Ex 2: Find the directional derivative of the function $f(x, y) = x^2y^3 - 4y$ at the point $(2, -1)$ in the direction of the vector $\vec{v} = < 2, 5 >$.

2. Directional Derivatives

Ex 3: Find the directional derivative $D_{\vec{u}}f(x, y)$ if $f(x, y) = x^3 - 3xy + 4y^2$ and $\vec{u}$ is the unit vector given by angle $\theta = \pi/6$. What is $D_{\vec{u}}f(1, 2)$?

2. Directional Derivatives

Note: Directional derivatives can be calculated for functions of 3 variables, but we can no longer thing of a graph.

Def: The directional derivative of f at (x_0, y_0, z_0) in the direction of a unit vector $\vec{u} =< a, b, c >$ is…

$$D_{\vec{u}} f(x_0, y_0, z_0) = \lim_{h \to 0} \frac{f(x_0 + ha, y_0 + hb, z_0 + hc) - f(x_0, y_0, z_0)}{h}$$

if this limit exists.

2. Directional Derivatives

Result: If f is a differentiable function of x, y, and z, then the directional derivative of f in the direction of the unit vector $\vec{u} =< a, b, c >$ is …

$$D_{\vec{u}}f(x, y, z) = \nabla f(x, y, z) \cdot \vec{u}$$

$$D_{\vec{u}}f(x_0, y_0, z_0) = \nabla f(x_0, y_0, z_0) \cdot \vec{u}$$

2. Directional Derivatives

<u>Ex 4</u>: If $f(x, y, z) = x \sin yz$,

a) find the gradient of f

b) find the directional derivative of f at $(1, 3, 0)$ in the direction of $\vec{v} = \vec{\imath} + 2\vec{\jmath} - \vec{k}$

3. Maximizing the Directional Derivative

Thm: Suppose f is a differentiable function of 2 or 3 variables. The maximum value of the directional derivative $D_{\vec{u}}f(\vec{x})$ is $|\nabla f(\vec{x})|$ and it occurs when $\vec{u}$ has the same direction as the gradient vector $\nabla f(\vec{x})$.

Why?

What about minimum value, 0 value?

3. Maximizing the Directional Derivative

Ex 5: Let $f(x, y) = xe^y$.

a) Find the rate of change of f at the point $P(2,0)$ in the direction from P to $Q\left(\frac{1}{2}, 2\right)$.

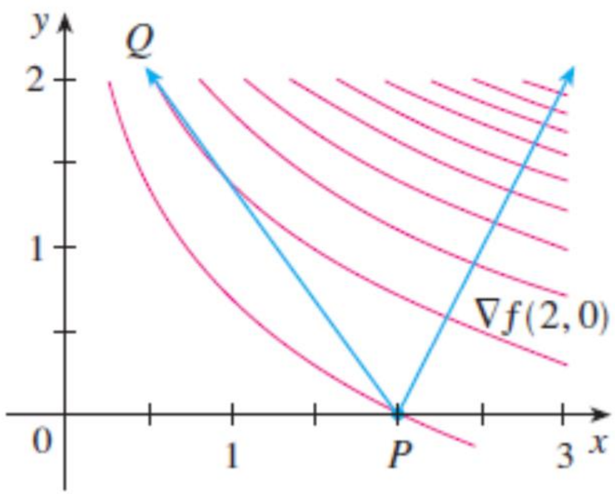
b) In what direction does f have the maximum rate of change? What is this maximum rate of change?

3. Maximizing the Directional Derivative

Ex 5: Let $f(x, y) = xe^y$.

At $(2, 0)$ the function in Example 6 increases fastest in the direction of the gradient vector $\nabla f(2, 0) = \langle 1, 2 \rangle$. Notice from Figure 7 that this vector appears to be perpendicular to the level curve through $(2, 0)$. Figure 8 shows the graph of f and the gradient vector.

Figure 7



3. Maximizing the Directional Derivative

Ex 6: Suppose that the temperature at a point (x, y, z) in space is given by $T(x, y, z) = \frac{80}{1+x^2+2y^2+3z^2}$, where T is measured in degrees Celsius and x, y, z in meters. In what direction does the temperature increase the fastest at the point $(1, 1, -2)$? What is the maximum rate of increase?

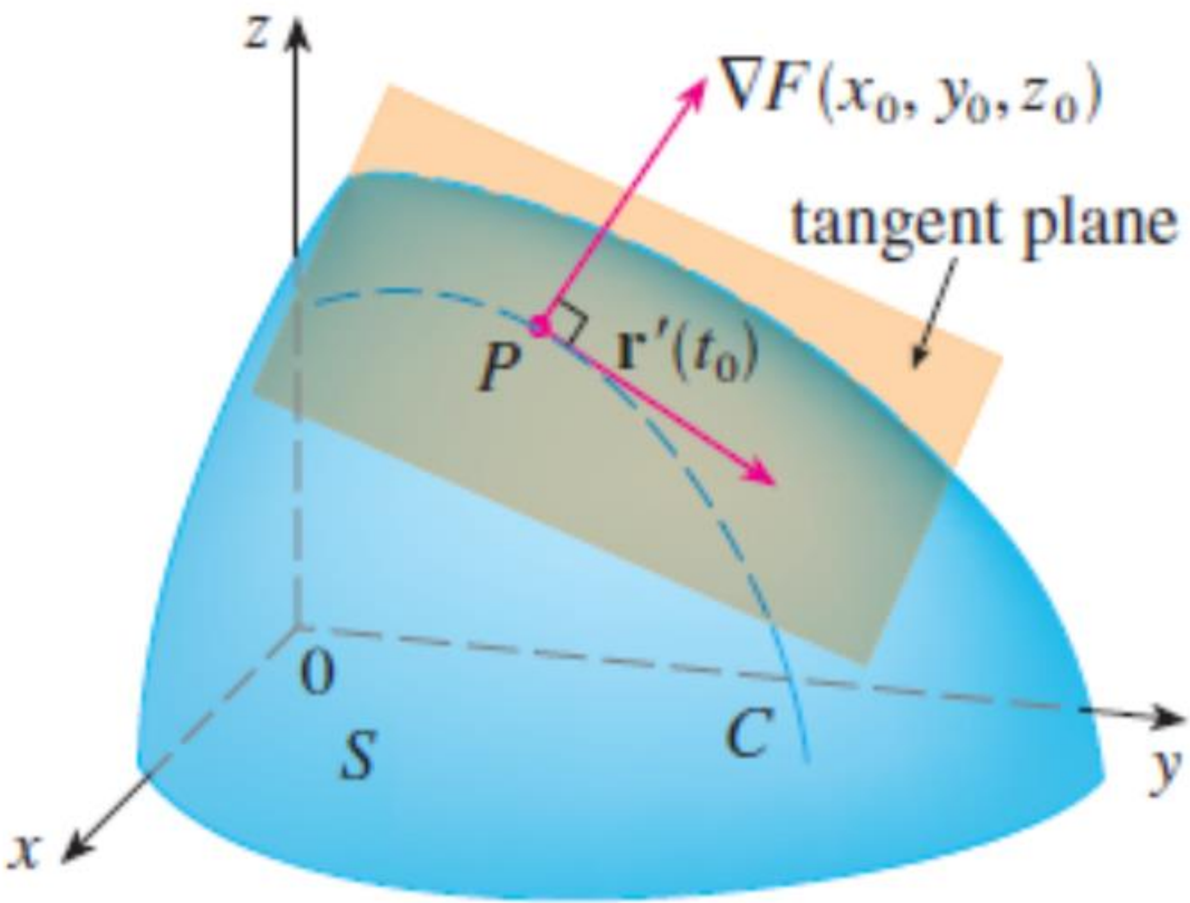
4. Tangent Planes to Level Surfaces

Situation:

- Let $F(x, y, z)$ be a function of 3 variables
- Consider the level surface $F(x, y, z) = k$ (call it S)
- Let $P(x_0, y_0, z_0)$ be a point on this surface
- Let $\vec{r}(t) =< x(t), y(t), z(t) >$ be a curve on this surface that passes through point P.
- Let t_0 be the parameter value corresponding to P. That is, $\vec{r}(t_0) =< x_0, y_0, z_0 >$.

4. Tangent Planes to Level Surfaces

Situation:



4. Tangent Planes to Level Surfaces

Tangent Plane:

Then the chain rule shows that the gradient of f is perpendicular to the tangent vector to the curve at P.

So the gradient vector is the normal vector of the tangent plane.

Eqn of Tangent Plane (to the level surface):

$$F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(x_0, y_0, z_0)(z - z_0) = 0$$

4. Tangent Planes to Level Surfaces

<u>The Normal Line:</u>

This is the line in the direction of the gradient (normal to the tangent plane).

<u>Eqn of Normal Line (to the level surface):</u>

$$\frac{x - x_0}{F_x(x_0, y_0, z_0)} = \frac{y - y_0}{F_y(x_0, y_0, z_0)} = \frac{z - z_0}{F_z(x_0, y_0, z_0)}$$

4. Tangent Planes to Level Surfaces

Ex 7: Find the equations of the tangent plane and normal line at the point $(-2, 1, -3)$ to the ellipsoid

$$\frac{x^2}{4} + y^2 + \frac{z^2}{9} = 3$$